A fractalization process for invariant curves in affine skew products of the plane

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Quasi-periodically forced maps are discrete dynamical systems of the form

$$\left.\begin{array}{rcl} \tilde{x} &=& f(x,\theta,\mu), \\ \tilde{\theta} &=& \theta+\omega, \end{array}\right\} \tag{1}$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{T}$, ω is irrational and μ is a real parameter. The map f is usually assumed to be of class C^r, $r \geq 1$. An invariant curve is the graph of a C^1 map $\theta \mapsto x(\theta)$ such that $f(x(\theta),\theta,\mu) = x(\theta+\omega)$. Assume that, for a given value of the parameter $\mu = \mu_0$, the system (1) has an attracting invariant curve, and that when μ goes from μ_0 to a critical value μ_1 this Lyapunov exponent goes to zero. We are interested in the possible behaviours of the invariant curve when μ approaches μ_1. In particular, we are interested in fractalization phenomena that might give rise to the appearance of a Strange Non-Chaotic Attractor.

To study this phenomenon we will focus on a simpler situation, given by the affine system

$$\left.\begin{array}{rcl} \tilde{x} &=& \mu A(\theta)x+b(\theta), \\ \tilde{\theta} &=& \theta+\omega, \end{array}\right\} \tag{2}$$

where ω is the golden mean, $x \in \mathbb{R}^2$ and, for each θ, $A(\theta)$ is a 2×2 real matrix and $b(\theta)$ a two-dimensional real vector. Moreover, we will assume that the corresponding linear system

$$\left.\begin{array}{rcl} \tilde{x} &=& \mu A(\theta)x, \\ \tilde{\theta} &=& \theta+\omega, \end{array}\right\}$$

is non-reducible due to a topological obstruction. A remarkable example of such a non-reducible system is given by

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We will prove that (2) has an invariant curve that displays a fractalization process when μ goes to a critical value.